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Non-perturbative corrections and modularity in $\mathcal{N} = 1$ type IIB compactifications

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Abstract: Non-perturbative corrections and modular properties of four-dimensional type IIB Calabi-Yau orientifolds are discussed. It is shown that certain non-perturbative α' corrections survive in the large volume limit of the orientifold and periodically correct the Kähler potential. These corrections depend on the NS-NS two form and have to be completed by D-instanton contributions to transform covariantely under symmetries of the type IIB orientifold background. It is shown that generically also the D-instanton superpotential depends on the two-form moduli as well as on the complex dilaton. These contributions can arise through theta-functions with the dilaton as modular parameter. An orientifold of the Enriques Calabi-Yau allows to illustrate these general considerations. It is shown that this compactification leads to a controlled four-dimensional $\mathcal{N} = 1$ effective theory due to the absence of various quantum corrections. Making contact to the underlying topological string theory the D-instanton superpotential is proposed to be related to a specific modular form counting D3, D1, D(-1) degeneracies on the Enriques Calabi-Yau.

Keywords: Superstring Vacua, String Duality, Flux compactifications, Supersymmetric Effective Theories.

Contents

1. Introduction

Recently much effort has focused on the study of orientifold compactifications of type II string theory with space-time filling D-branes and background fluxes. The reason is that these compactifications can lead to calculable four-dimensional effective theories supporting string vacua relevant for particle physics and cosmology [1 − 3]. Particularly well controlled are warped type IIB Calabi-Yau orientifolds with space-time filling D3 and D7 branes which yield a four-dimensional effective theory with $\mathcal{N} = 1$ supersymmetry [4, 3]. It was realized that in these compactifications the inclusion of background fluxes and certain non-perturbative corrections might lead to a stabilization of all unwanted scalar moduli fields in a local vacuum [5]. This was demonstrated for specific examples e.g. in refs. [6 − 9] and strengthened the believe in a vast landscape of supersymmetric and non-supersymmetric string vacua [3]. In order to study these vacua a precise knowledge of the $\mathcal{N} = 1$ characteristic data of the four-dimensional effective theory is of central importance. In particular, this includes the understanding of perturbative and non-perturbative corrections to the Kähler potential and the superpotential.

The aim of this work is to investigate the leading perturbative and non-perturbative corrections for Calabi-Yau orientifolds with O3 and O7 planes. We first study the α' corrections inherited from the underlying $\mathcal{N} = 2$ theory which survive the large volume limit of the orientifold. This includes the perturbative α' corrections discussed in ref. [10]. Moreover, we argue by using the results of refs. [11, 12] that also non-perturbative α' corrections involving the NS-NS B-field can survive the large volume limit of the orientifold. These corrections are generically present in compactifications in which the B-field is not entirely projected out by the orientifold symmetry.[1] The real B-field scalars combine with the scalars of the R-R two-form C_2 into complex scalars G^a through the combination $C_2 - \tau B_2$, where τ is the complex dilaton-axion [11]. The perturbative and non-perturbative α' corrections in the orientifold large volume limit do not correct the $\mathcal{N} = 1$ coordinates. They do however contribute to the Kähler potential and we will be able to determine these corrections explicitly in terms of the topological invariants of the underlying Calabi-Yau manifold. We will also study the non-perturbative superpotential generated by D3-instantons wrapping a four-cycle in the Calabi-Yau manifold and show that it generically depends on the scalars τ and G^a. In order to do that, we implement the non-perturbative symmetries inherited from the full type IIB string theory.

Type IIB string theory possesses a strong-weak duality known as S-duality. This non-perturbative symmetry relates one type IIB theory with complex string coupling τ to a dual type IIB string theory with string coupling $-1/\tau$. Moreover, it exchanges the NS-NS and R-R two-forms and thus fundamental strings with D1 branes. Together with the shifts in the axion, $\tau \to \tau + 1$, the S-duality transformation generates the discrete duality group $Sl(2, \mathbb{Z})$. In an $\mathcal{N} = 1$ compactification this group will generically be reduced further or broken completely by to the non-trivial background geometry. However, in the orientifold compactifications under consideration the complex dilaton τ does not vary over the compact six-dimensional geometry and appears as four-dimensional chiral field [4, 3]. In this limit we expect that a subgroup Γ_S of the full $Sl(2, \mathbb{Z})$ duality is a symmetry of the four-dimensional theory in analogy to refs. [13, 14]. Determining the transformations of the $\mathcal{N} = 1$ coordinates under Γ_S as well as integral shifts of the NS-NS B-field allows us to study the moduli dependence and symmetries of the Kähler potential and superpotential in the orientifold large volume limit.

We begin by discussing the transformation properties of the Kähler potentials under Γ_S when α' corrections are included. In order for these to be invariant under Γ_S also contributions from D1 and D(-1) branes have to be taken into account. In general, it is hard to compute these corrections. We will however be able to discuss candidate completions which reproduce the perturbative and non-perturbative α' corrections and admit the desired transformation properties. In order to obtain these solutions we will simply sum over images of the α' corrections under the duality group following [15, 16]. This does however not guarantee that the result is the true non-perturbative completion. Firstly, this analysis is only valid in the orientifold limit in which the type IIB symmetry is not entirely broken by the vacuum and a discrete group Γ_S is preserved. Secondly, even though this symmetry group ideally restricts the answer to be generated by a finite set of appropriately transforming functions additional boundary conditions are needed to fix the precise form

[1] An example of a Calabi-Yau orientifold with non-vanishing B-field moduli is presented in the second part of this paper. For other examples which admit these additional moduli fields, see e.g. ref. [9].

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of the duality invariant completion.[2] For corrections to the $\mathcal{N}=1$ Kähler potential this task is even more involved, since the Kähler potential is not protected by holomorphicity or non-renormalization theorems. The application of string-string dualities such as heterotic-F-theory duality might however help to compute these corrections explicitly as argued, for example, in refs. [21, 22]. One expects that modularity arguments are however more powerful when arguing about the superpotential.

In $\mathcal{N}=1$ theories the superpotential is holomorphic and protected against perturbative corrections. For the type IIB orientifold setups the determination of the D3-instanton superpotential is of central importance. However, its explicit form is in general hard to determine [23 – 25]. Nevertheless, by combining holomorphicity and modular properties under the inherited type IIB $Sl(2,\mathbb{Z})$ symmetry as well as shifts in the NS-NS B-field the moduli dependence of the superpotential in general large volume orientifolds can be discussed. In case the complex dilaton τ varies over the internal space only a local analysis of the superpotential can be performed [26, 27]. Here our results are more restrictive due to the fact that τ, G^a do not vary over the compact space. We find that the complex fields G^a depending on the NS-NS and R-R two-form moduli naturally arise through products of theta-functions and modular forms with the complex dilaton-axion τ as modular parameter. In the second part of the paper we propose that this set of theta-functions can be determined for a specific orientifold example.

The specific example we consider is an orientifold of the Enriques Calabi-Yau. The underlying Calabi-Yau manifold is a $K3$ fibration of the form $Y_E = (K3 \times T^2)/\mathbb{Z}_2$ [28, 29], where the freely acting $\mathbb{Z}_2$ symmetry yield a minus sign on the complex coordinate of T^2 and acts as the Enriques involution on the $K3$ surface [30]. We will show that an appropriate definition of the orientifold projection allows to explicitly determine the $\mathcal{N}=1$ four-dimensional effective theory. Since the geometric moduli space of the underlying $\mathcal{N}=2$ theory is not corrected by world-sheet instantons or perturbative α' corrections the resulting $\mathcal{N}=1$ theory is particularly well controlled. We will show that the $\mathcal{N}=1$ moduli space is a product of two cosets $\tilde{\mathcal{M}}_{\rm sk} \times \tilde{\mathcal{M}}_{\rm q}$. The first factor $\tilde{\mathcal{M}}_{\rm ks}$ arises from the reduction of the $\mathcal{N}=2$ special Kähler manifold containing the complex structure deformations of Y_E. It is itself a special Kähler manifold and was studied intensively in the literature [31, 20]. The reduction of the $\mathcal{N}=2$ quaternionic manifold leads to a Kähler manifold $\tilde{\mathcal{M}}_{\rm q}$ of half its dimension. Remarkably, $\tilde{\mathcal{M}}_{\rm q}$ can be identified with the original $\mathcal{N}=2$ special Kähler manifold of complexified Kähler structure deformations $\mathcal{M}_{\rm ks}$ times an $Sl(2,\mathbb{R})/\,\mathrm{U}(1)$ factor. In this identification half of the NS-NS fields arising as real parts of coordinates on $\mathcal{M}_{\rm sk}$ are replaced by R-R fields. The resulting $\mathcal{N}=1$ coordinates encode the correct couplings to D(-1), D1 and D3 branes. Note however, that this duality is not performed in the large volume coordinates on $\mathcal{M}_{\rm sk}$, but rather at a special locus where also the volume of the K3 fiber can be small.

The physics in the regime where the K3 fiber of the Enriques Calabi-Yau is small was studied intensively in the underlying $\mathcal{N}=2$ theory. It was shown in ref. [32] that at the limit were the K3 fiber is of Planck length the type II theory undergoes a phase

[2]See [17 – 20] for the discussion of an analogous problem within topological string theory.

transition somewhat similar to the well-known conifold transition. It was later argued in ref. [31] that the light BPS degrees of freedom at this locus are bound states of D4, D2 and D0 branes wrapped around specific four and two-cycles of Y_E. The authors of [31] showed that the topological string theory on the Enriques Calabi-Yau can be resummed to count the degeneracies of these degrees of freedom. The leading contributions arise through a particular holomorphic function $\Phi_{\rm B}$ known from the work of Borcherds [33, 34] and Harvey, Moore [35, 36]. Here we will employ the duality of the theory on $\mathcal{M}_{\rm ks}$ at this special locus to the corresponding orientifold theory. We propose that $\Phi_{\rm B}$ naturally arises in the $\mathcal{N}=1$ superpotential containing the D3-instanton corrections proportional to e^{iT_S}, where T_S contains the volume of the K3 fiber. In accord with our general considerations, the coefficients are indeed generalizations of theta functions depending on the modular parameter τ, the dilaton-axion, as well as the scalars G^a arising from the NS-NS and R-R two-forms. The study of the Enriques orientifold exemplifies nicely the interplay of holomorphicity and symmetry properties for the non-perturbative superpotential.

This paper is organized as follows. In section 2.1 we briefly review the effective theory of type IIB orientifolds with O3 and O7 planes. We discuss the reduction of an $\mathcal{N}=2$ theory defined by two general pre-potentials for complex structure and Kähler structure deformations respectively. It is then shown in section 2.2 that certain α' corrections survive in the large volume limit of the orientifold and correct the Kähler potential in an explicitly calculable way. The modular completion of these corrections by D(-1) and D1 brane contributions is discussed in section 2.3. In section 2.4 we turn to the discussion of the non-perturbative superpotential generated by D3-instantons. We study its transformations under the type IIB symmetries and argue for a moduli dependence through generalizations of theta functions. In section 3 we present an explicit example by introducing an orientifold of the Enriques Calabi-Yau manifold. We first summarize some details about the $\mathcal{N}=2$ theory in section 3.1. The Kähler potential and an interesting duality map is studied in 3.2. Finally, in section 3.3 we propose a particular non-perturbative superpotential counting degeneracies of D3, D1, D(-1) bound states.

2. Non-perturbative corrections and modularity

In this section we discuss non-perturbative corrections and the transformation properties of the $\mathcal{N}=1$ effective action of type IIB string theory compactified on an orientifold background. We begin with a brief review of the four-dimensional effective theory in section 2.1. In section 2.2 we show that in the orientifold large volume limit the perturbative and certain non-perturbative α' corrections inherited from the underlying $\mathcal{N}=2$ theory correct the $\mathcal{N}=1$ Kähler potential. We will argue that these corrections generically do not respect the type IIB $Sl(2,\mathbb{Z})$ symmetries in section 2.3. Since in the orientifold limit a subgroup Γ_S of this symmetry group is expected to be preserved we comment on modular completions of the Kähler potential. Finally, in section 2.4 we analyze the transformation properties of the $\mathcal{N}=1$ complex coordinates and constrain the D-instanton superpotentials to contain generalizations of theta functions. This leads to a new moduli dependence of the superpotential which is generic for many orientifold compactifications.

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2.1 Brief review of the effective action of type IIB orientifolds

In this section we review the $\mathcal{N}=1$ effective supergravity theory arising by compactification of type IIB supergravity on an orientifold background following [4, 10 − 12, 37]. We will focus on orientifold projections yielding O3 and O7 planes and include the leading perturbative α' corrections [10] as well as the world-sheet instanton corrections inherited from the underlying $\mathcal{N}=2$ theory [12]. Since there exists a number of reviews [3] on this topic we will keep our discussion brief.

In type IIB orientifolds with O3/O7 planes the orientifold projection takes the form $(-1)^{F_L}\Omega_p\sigma$, where F_L is the left fermion number, Ω_p is the world-sheet parity reversal and σ is some geometric involutive symmetry of the background. In order to preserve $\mathcal{N}=1$ supersymmetry σ has to be a holomorphic and isometric involution. It acts non-trivially on the internal Calabi-Yau manifold Y and leaves the four flat directions invariant. For models with O3/O7 planes σ acts on the Kähler form J and holomorphic three form Ω of Y as

$$\sigma^* J = J\,, \qquad \sigma^*\Omega = -\Omega\,, \tag{2.1}$$

where σ^* is the pull-back. In order to remain in the spectrum the NS-NS and R-R fields have to transform as follows under σ^*. The dilaton ϕ, the axion C_0 as well as the four-form C_4 are invariant under the action of σ, while the NS-NS two-form B_2 and R-R two-form C_2 transform with a minus sign. Type IIB Calabi-Yau orientifolds with O3/O7 planes have the following truncated $\mathcal{N}=1$ moduli space:

$$\tilde{\mathcal{M}}_{\rm sk} \times \tilde{\mathcal{M}}_{\rm q}\,, \tag{2.2}$$

where $\tilde{\mathcal{M}}_{\rm sk}$ is a special Kähler manifold inside the $\mathcal{N}=2$ special Kähler manifold $\mathcal{M}_{\rm sk}$ and $\tilde{\mathcal{M}}_{\rm q}$ is a Kähler manifold inside the $\mathcal{N}=2$ quaternionic manifold $\mathcal{M}_{\rm q}$. In the following we will describe the geometry of the moduli space (2.2) in more detail.

Let us start with some comments on the cohomology of the orientifold theory and the reduction of $\mathcal{M}_{\rm sk}$. Since σ is a holomorphic involution the cohomology groups $H^{(p,q)}$ split into two eigenspaces under the action of σ^* as $H^{(p,q)} = H^{(p,q)}_+ \oplus H^{(p,q)}_-$. We denote the dimensions of $H^{(p,q)}_\pm$ by $h^{(p,q)}_\pm$. The four-dimensional invariant spectrum is found by using a Kaluza-Klein expansion in harmonic forms keeping only the fields which in addition obey the correct transformations under σ^*. This induces a reduction of the special Kähler manifold $\mathcal{M}_{\rm sk}$ for the orientifold setups. Since σ transforms the complex three-form Ω with a minus sign the complex structure deformations parametrized by the elements of $H^{(2,1)}$ are reduced to $h^{(2,1)}_-$ complex scalars z^k. It can be shown that these define a $h^{(2,1)}_-$ dimensional special Kähler submanifold $\tilde{\mathcal{M}}_{\rm sk}$ of the original $\mathcal{N}=2$ moduli space of complex structure deformations. The Kähler potential on $\tilde{\mathcal{M}}_{\rm sk}$ takes the well-known form

$$K_{\rm cs}(z,\bar z) = -\ln\big[i\int_Y \Omega(z)\wedge\bar\Omega(\bar z)\big]\,, \tag{2.3}$$

where $\Omega(z^k)$ varies holomorphically over $\tilde{\mathcal{M}}_{\rm sk}$. Recall that in the underlying $\mathcal{N}=2$ theory the complex scalars z were part of vector multiplets. In the orientifold reduction also $h^{(2,1)}_+$ of the vectors survive. The gauge-kinetic coupling function is the second derivative of the pre-potential of the underlying $\mathcal{N}=2$ special Kähler manifold $\mathcal{M}_{\rm sk}$ with respect to the complex structure deformations z^κ, which are then set to zero in the orientifold scenario [11].

The reduction of the quaternionic space $\mathcal{M}_{\rm q}$ is slightly more involved. Since σ leaves the Kähler form J invariant and yields a minus sign on the B_2 field we expand

$$J = v^\alpha \omega_\alpha\,, \quad \alpha = 1,\ldots,h_+^{(1,1)}\,, \qquad\qquad B_2 = b^a \omega_a\,, \quad a = 1,\ldots,h_-^{(1,1)}\,, \tag{2.4}$$

where ω_α is an integral basis of $H^2_+(Y,\mathbb{Z})$ and ω_a is an integral basis of $H^2_-(Y,\mathbb{Z})$. The conditions (2.4) defines a real subspace of the $h^{(1,1)}$ dimensional space of complexified Kähler deformations $\mathcal{M}_{\rm ks}$ of Y. This is due to the fact that either the real or the complex part of the complexified Kähler form survives:

$$-B_2 + iJ = t^A \omega_A = -b^a \omega_a + i v^\alpha \omega_\alpha\ . \tag{2.5}$$

Let us now include the R-R forms. Invariance under the orientifold projection enforces the expansions

$$C_2 = c^a \omega_a\,, \qquad\qquad C_4 = \rho_\alpha \tilde\omega^\alpha\,, \tag{2.6}$$

where ω_a was already introduced in (2.4) and we have denoted by $\tilde\omega^\alpha$ an integral basis of $H^4_+(Y,\mathbb{Z})$ dual to ω_α. Note that in (2.6) we have only displayed the part of the expansion of C_4 which leads to four-dimensional scalars.[3] Let us now define the even form

$$\rho = 1 + t^A\omega_A - \mathcal{F}_A \tilde\omega^A + (2\mathcal{F} - t^A \mathcal{F}_A)\epsilon\,, \tag{2.7}$$

where $\mathcal{F}$ is the pre-potential on $\mathcal{M}_{\rm ks}$ and $\mathcal{F}_A$ is its first derivative with respect to t^A. The orientifold effective theory including a general pre-potential $\mathcal{F}$ was derived in refs. [12, 37]. It was shown there, that the complex coordinates on the Kähler manifold $\tilde{\mathcal{M}}_{\rm q}$ are obtained in the expansion

$$\rho_c \equiv e^{-B_2}\wedge C^{\rm RR} + i{\rm Re}\big(C\rho\big) = \tau + G^a\omega_a - T_\alpha \tilde\omega^\alpha\,, \tag{2.8}$$

where $C^{\rm RR} = C_0 + C_2 + C_4$ and the function C is identified with the dilaton $e^{-\phi}$. The Kähler potential for the complex scalars τ, G^a, T_α is then shown to be

$$\begin{aligned} K_{\rm q}(\tau,G,T) &= -2\ln\left[i\int_Y \langle C\rho, \overline{C\rho}\rangle\right] \\ &= -2\ln\left[i|C|^2\big(2(\mathcal{F}-\bar{\mathcal{F}}) - (\mathcal{F}_\alpha + \bar{\mathcal{F}}_\alpha)(t^\alpha - \bar t^\alpha)\big)\right], \end{aligned} \tag{2.9}$$

where we have inserted the even form ρ defined in (2.7) to evaluate the second equality.[4] Note that K is a function of the imaginary part ${\rm Im}\rho_c = {\rm Re}(C\rho)$ of ρ_c only. This implies that K only depends on the combinations $\tau - \bar\tau$, $G^a - \bar G^a$ and $T_\alpha - \bar T_\alpha$. For a general pre-potential $\mathcal{F}$ it is impossible to explicitly write K as the function of τ, G^a, T_α. This is due to

[3]The vectors discussed in the previous paragraph arise precisely in the expansion of C_4 into appropriate three-forms.

[4]The anti-symmetric product between two even forms ρ,λ is defined as the alternating wedge product $\langle\rho,\lambda\rangle = \rho_0\wedge\lambda_6 - \rho_2\wedge\lambda_4 + \rho_4\wedge\lambda_2 - \rho_6\wedge\lambda_0$, where ρ_p, λ_p are the p-form parts of ρ,λ.

the fact that one would need to express $\mathrm{Im}(C\rho)$ as a function of $\mathrm{Im}\rho_c = \mathrm{Re}(C\rho)$ appearing in the $\mathcal{N} = 1$ coordinates (2.8). This functional dependence is highly non-polynomial and can only be determined explicitly in specific examples.[5] Nevertheless, one can derive the Kähler metric by using the underlying $\mathcal{N} = 2$ special geometry [12] or the work of Hitchin [38] as done in [39].

So far we have determined the $\mathcal{N} = 1$ kinetic terms of the scalar and vector fields. Masses for these scalar fields can be generated by a non-trivial superpotential or the presence of D-terms. In the rest of the paper we will only discuss the inclusion of a superpotential. In type IIB orientifolds with O3/O7 planes it can be generated by non-vanishing R-R and NS-NS three-form flux F_3 and H_3 as well as non-perturbative corrections due to D-instantons. It takes the form [23, 40, 4, 5]

$$W = \int_Y \Omega(z) \wedge \big(F_3 - \tau H_3\big) + W_{\text{D-inst}}(\tau, z, G, T, \dots) \ . \tag{2.10}$$

The first term is the well-known Gukov-Vafa-Witten flux superpotential, while the second term encodes the D-instanton effects. We will discuss the field dependence and modular properties of $W_{\text{D-inst}}$ in section 2.4. In order to do that it is often convenient to also refer to the underlying F-theory description of the orientifold setup. We therefore end this section with some remarks on the F-theory embedding and four-dimensional symmetries.

Type IIB orientifolds with O3 and O7 planes arise as a special limit of F-theory [41] compactified on particular four-dimensional Calabi-Yau manifolds [42]. These fourfolds have to admit an elliptic fibration

$$Y_4 \to B_3 \,, \tag{2.11}$$

where B_3 is some three-dimensional base manifold. The complex structure of the torus fiber corresponds to the complex dilaton τ introduced above. In general τ can vary over the base B_3. This implies the existence of a modular group Γ_M associated to the elliptic fibration. This group encodes the monodromies around the singular points of the fibration and is a discrete subgroup of the torus symmetry group $Sl(2,\mathbb{Z})$. The complete $Sl(2,\mathbb{Z})$ symmetry corresponds to the non-perturbative symmetry of type IIB string theory. In the full F-theory compactification it is reduced or broken due to the background geometry Y_4 [41, 43]. Roughly speaking, the larger the modular group $\Gamma_M \in Sl(2,\mathbb{Z})$, the fewer symmetries survive in the effective four-dimensional action.

In this paper we will entirely focus on the orientifold limit reviewed in this section [4, 3]. It was shown in [42] that in this limit the base B_3 can be obtained as a quotient of a Calabi-Yau manifold by an involution σ as discussed above. The singularities of elliptic fibration (2.11) determine the location of the space-time filling O7 planes and D7 branes. However, in the above orientifold limit, both the complex dilaton as well as the fields G^a do not vary over the base B_3, but correspond to chiral fields in four space-time dimensions. In other words, in this limit the monodromy group Γ_M acts trivially on τ, G^a and we expect that a subgroup $\Gamma_S \subset Sl(2,\mathbb{Z})$ survives as a symmetry of the effective action. This symmetry posses stringent constrains on the $\mathcal{N} = 1$ characteristic data of the orientifold

[5]This is equivalent to the problem of solving the attractor equations for $\mathcal{N} = 2$ black holes.

compactification in analogy to [13, 14]. In the next sections we discuss these conditions in detail. Clearly, a more general analysis would consider the full F-theroy compactification and we hope to return to this problem in forthcoming work. Let us just remark here, that there is no known effective action of twelve-dimensional F-theory. The four-dimensional $\mathcal{N} = 1$ effective theory thus has to be determined by an M-theory lift. More precisely, one compactifies M-theory on the elliptically fibered fourfold Y_4 to obtain a three-dimensional effective theory. This theory is then lifted to four-dimensions by growing an extra non-compact dimension. The F-theory moduli thus arise from the expansion of the M-theory fields, such as the three-form C_M, into harmonics of Y_4. A detailed discussion of the derivation of the effective action can be found, for example, in refs. [44, 45, 37].

2.2 Perturbative and non-perturbative α' corrections in the orientifold large volume limit

In this section we simplify the discussion and work in the large volume limit of the orientifold Y/σ. This implies that we consider the regime where v^α is large. Note that this is not the same as demanding that all v^A are large on the underlying Calabi-Yau manifold. To make this more precise, recall that the orientifold projection forces us to set $v^a = 0$ for $h^{(1,1)}_-$ directions and $b^\alpha = 0$ for $h^{(1,1)}_+$ directions. Together, these conditions define a real $h^{(1,1)}$-dimensional Lagrangian submanifold in the complex $\mathcal{N} = 2$ Kähler moduli space [12]. This subspace does not necessarily contain the $\mathcal{N} = 2$ large volume point. Examples for such a situation can be found in ref. [46]. Clearly, in case the $\mathcal{N} = 2$ large volume point is not in the $\mathcal{N} = 1$ locus, the orientifold projection does not commute with the taking all two-volumes to be large and we are forced to include instanton corrections to the classical orientifold set-up.

A possible point where those corrections become relevant are the contributions depending on $t^a = -b^a$. World-sheet instantons coupling via the exponential e^{it^a} are not necessarily suppressed in the large volume limit of the orientifold. We therefore include the non-pertubative α' corrections inherited from the underlying $\mathcal{N} = 2$ theory. More precisely, we obtain in this limit a pre-potential of the form[6]

$$\begin{aligned} \mathcal{F} &= \mathcal{F}_{\text{class}} + \mathcal{F}_{\text{pert}} + \mathcal{F}_{\text{b}} \\ &= -\frac{1}{3!}\mathcal{K}_{ABC}t^A t^B t^C - \frac{i}{2}\zeta(3)\chi + i \sum_{\beta \in H_2^-(Y,\mathbb{Z})} n^0_\beta \, \mathrm{Li}_3(e^{ik_a t^a}) \, , \end{aligned} \tag{2.12}$$

where $k_a = \int_\beta \omega_a$ with ω_a being an integral basis of $H^2_-(Y,\mathbb{Z})$. Let us discuss the three contributions in (2.12) in turn. The cubic term $\mathcal{F}_{\text{class}}$ corresponds to the classical contribution and we denote the triple intersections of the integral basis $\omega_A \in H^2(Y,\mathbb{Z})$ by

$$\mathcal{K}_{ABC} = \int_Y \omega_A \wedge \omega_B \wedge \omega_C \ . \tag{2.13}$$

[6] Note that in general $\mathcal{F}$ can also admit a cubic and linear term of the form $B_{AB}t^A t^B$, $A_A t^A$. However, since A_A, B_{AB} are always real it is easy to check that they do not appear in the Kähler potential (2.9). They only correct the coordinates T_α and we will not consider these contributions in the following.

Note that in the orientifold setup consistency requires that for the spilt $\omega_A = (\omega_\alpha, \omega_a)$ the following intersections have to vanish:

$$\mathcal{K}_{\alpha\beta a} = \mathcal{K}_{abc} = 0 \ . \tag{2.14}$$

In other words only the intersections $\mathcal{K}_{\alpha\beta\gamma}$ and $\mathcal{K}_{\alpha ab}$ with zero or two negative indices can appear in (2.12). The second term $\mathcal{F}_{\text{pert}}$ in (2.12) is proportional to the Euler characteristic $\chi = 2(h^{(1,1)} - h^{(2,1)})$ of Y. It corresponds to an $(\alpha')^3$ perturbative correction of the effective action and was first considered in orientifold setups in ref. [10].

The third term $\mathcal{F}_{\text{b}}$ is inherited from the non-perturbative α' corrections of the $\mathcal{N} = 2$ pre-potential and was not discussed in the literature so far. In the large volume limit of the orientifold only the terms depending on the B-field moduli $t^a = -b^a$ survive in the third polylogarithm $\text{Li}_3(x) = \sum_{n>0} n^{-3} x^n$. All other contributions are suppressed exponentially by the volume of the curves in $H_2^+(Y,\mathbb{Z})$. In other words, only the terms proportional to the integer genus zero Gopakumar-Vafa invariants n_β^0 [47] for a curve β in the negative eigenspace $H_2^-(Y,\mathbb{Z})$ remain in the pre-potential. They can be determined for many explicit examples of Calabi-Yau manifolds my using mirror symmetry [48]. However, we have to make a cautionary remark on the convergence of the expansion (2.12). Since the polylogarithm $\text{Li}_3(e^{ik_a t^a})$ is bounded $\mathcal{F}_{\text{b}}$ appears divergent when summing over all β. This would be very generically the case if β is not restricted to any sublattice in $H_2(Y,\mathbb{Z})$ since the Gopakumar-Vafa invariants grow very rapidly. However, in the expression (2.12) for $\mathcal{F}_{\text{b}}$ we only sum over degrees k_A which are of the form $k_A = (0, k_a)$, i.e. vanish on the positive eigenspace of the orientifold. There are indeed examples for which the n_β^0 truncates on such a sublattice $(0, k_a)$.[7] More generally, in case $\mathcal{F}_{\text{b}}$ is not finite this can be traced back to the fact that we are actually working in the wrong coordinates t^a. Before restricting to the orientifold limit $\text{Im}t^a \to 0$ the expression $\mathcal{F}_{\text{b}}$ has to be resummed in terms of dual coordinates valid around $\text{Im}t^a = 0$. One is then able to implement the orientifold projection with a finite $\mathcal{F}_{\text{b}}$. In the following we will simply assume that $\mathcal{F}_{\text{b}}$ is finite when restricting our general considerations to appropriate specific examples.

In order to determine the $\mathcal{N} = 1$ coordinates we first insert the large volume pre-potential (2.12) into the definition (2.7) of the even form ρ. Due to the presence of the α' corrections $\mathcal{F}_{\text{pert}} + \mathcal{F}_{\text{b}}$ the classical expression $\rho_{\text{class}} = e^{-B_2 + iJ}$ will receive non-trivial corrections. However, it is easy to check that these corrections will not contribute to the definition of the $\mathcal{N} = 1$ coordinates τ, G^a, T_α defined in (2.8). A straightforward computation shows that τ, G^a, T_α are given in terms of the real coordinates introduced in (2.4) and (2.6) by

$$\tau = C_0 + ie^{-\phi} \, , \qquad\qquad G^a = c^a - \tau b^a \, , \tag{2.15}$$

$$T_\alpha = \frac{1}{2} ie^{-\phi} \mathcal{K}_{\alpha\beta\gamma} v^\beta v^\gamma - \tilde{\rho}_\alpha - \frac{1}{2(\tau - \bar{\tau})} \mathcal{K}_{\alpha ab} G^a (G - \bar{G})^b \, , \tag{2.16}$$

where $\tilde{\rho}_\alpha = \rho_\alpha - \frac{1}{2}\mathcal{K}_{\alpha ab} c^a b^b$. These are precisely the coordinates introduced in ref. [11].[8]

[7] We are grateful to A. Klemm for discussions on this point.

[8] In contrast to ref. [11] we rescaled the coordinates $T_\alpha = \frac{2i}{3} T_\alpha^{\text{ref.}}$ and identified $\tilde{\rho}_\alpha = \rho_\alpha^{\text{ref.}}$.

However, in contrast to the classical results the Kähler potential K_{q} is now corrected by the α' contributions encoded by $\mathcal{F}_{\mathrm{pert}} + \mathcal{F}_{\mathrm{b}}$ in (2.12).

Let us make this more precise and evaluate the Kähler potential for the large volume pre-potential (2.12). Inserting $\mathcal{F}$ into the general expression (2.9) for K_{q} one derives

$$K_{\mathrm{q}} = -2\ln\Big[e^{-2\phi}\Big(\frac{1}{3!}\mathcal{K}_{\alpha\beta\gamma}v^\alpha v^\beta v^\gamma + 2\zeta(3)\chi - 4\mathrm{Im}\mathcal{F}_{\mathrm{b}}\Big)\Big]\,. \tag{2.17}$$

In this expression the non-perturbative corrections inherited from the underlying $\mathcal{N}=2$ theory take the form

$$\begin{aligned} \mathrm{Im}\mathcal{F}_{\mathrm{b}}(\tau,G) &= \frac{1}{2}\sum_{\beta\in H_2^-(Y,\mathbb{Z})} n_\beta^0\Big[\mathrm{Li}_3\Big(e^{i\frac{k_a(G^a-\bar{G}^a)}{\tau-\bar{\tau}}}\Big) + \mathrm{Li}_3\Big(e^{-i\frac{k_a(G^a-\bar{G}^a)}{\tau-\bar{\tau}}}\Big)\Big]\,,\\ &= \sum_{\beta\in H_2^-(Y,\mathbb{Z})}\sum_{n=1}^{\infty}\frac{n_\beta^0}{n^3}\,\cos\Big(n\frac{k_a(G^a-\bar{G}^a)}{\tau-\bar{\tau}}\Big)\,, \end{aligned} \tag{2.18}$$

where $k_a = \int_\beta \omega_a$ as in (2.12). This implies that the moduli dependence on τ, G^a of both α' corrections to the Kähler potential can be determined explicitly. Rescaling the Kähler deformations v^α to the Einstein frame we can write K_{q} into the form

$$K_{\mathrm{q}} = -\ln\big[-i(\tau-\bar{\tau})\big] - 2\ln\Big[V_E + \frac{1}{(2i)^{3/2}}\big(\tau-\bar{\tau}\big)^{3/2}\big[2\zeta(3)\chi - 4\mathrm{Im}\mathcal{F}_{\mathrm{b}}\big]\Big]\,, \tag{2.19}$$

where $V_E(\tau,G,T)$ is the Einstein frame volume of the Calabi-Yau orientifold and $\mathcal{F}_{\mathrm{b}}(\tau,G)$ is explicitly given in (2.18). The large volume Kähler potential (2.19) includes the special cases derived in refs. [4, 10, 11]. Here we were able to include the non-perturbative contribution $\mathcal{F}_{\mathrm{b}}(\tau,G)$ and have shown that they can be expressed as explicit functions in $G^a - \bar{G}^a$ and $\tau - \bar{\tau}$. In the next section we will discuss the invariance of the general Kähler potential (2.19) under the $Sl(2,\mathbb{Z})$ symmetry of type IIB string theory as well as shifts in the B-field.

2.3 Symmetries of the Kähler potential

In this section we discuss the transformation properties of the Kähler potential under dualities inherited from the ten-dimensional type IIB string theory. We will focus on the $Sl(2,\mathbb{Z})$ symmetry of type IIB as well as shifts in the NS-NS two-form B_2.

Let us begin by discussing the symmetry of K under shifts of the NS-NS two-form B_2. More precisely, we will consider

$$B_2 \quad\to\quad B_2 + 2\pi\chi_2\,, \qquad\qquad \chi_2 = n^a\omega_a\,, \tag{2.20}$$

where χ_2 is an integral two form in $H^2_-(Y_E,\mathbb{Z})$. For this transformation we easily verify that the Kähler potential is invariant. The Einstein frame volume V_E in (2.19) is invariant due to its purely geometrical origin, while the perturbative contribution from $\mathcal{F}_{\mathrm{pert}}$ is independent of B_2 and hence trivially invariant. Only the non-perturbative corrections encoded by $\mathcal{F}_{\mathrm{b}}$ explicitly depend on B_2. However, B_2 only arises through the exponential $\exp(-i\int_\beta B_2)$

which is invariant under integral shifts. We thus conclude that K is indeed invariant under (2.20). In contrast, we will see in the next section that the $\mathcal{N}=1$ coordinates G^a, T_α transform non-trivially under the shifts (2.20). This will allow us to infer valuable information about the moduli dependence of the D-instanton superpotential in (2.10).

Let us turn to the symmetry inherited from the underlying type IIB theory. Recall that type IIB string theory admits the discrete symmetry group $Sl(2,\mathbb{Z})$. Denoting the ten-dimensional dilaton-axion as $\tau = C_0 + ie^{-\phi}$ this group acts by modular transformations and rotates the ten-dimensional NS-NS and R-R two-forms B_2 and C_2 into each other. More explicitly, we have

$$\tau \quad \rightarrow \quad \frac{a\tau+b}{c\tau+d}\,, \qquad\qquad \begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} a\,C_2 + b\,B_2 \\ c\,C_2 + d\,B_2 \end{pmatrix}\,, \tag{2.21}$$

where the integer matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an element of $Sl(2,\mathbb{Z})$.[9] These transformations include in particular the map $\tau \rightarrow -1/\tau$ which inverts the string coupling and corresponds to the strong-weak duality known as S-duality. Compactifying type IIB string theory on a Calabi-Yau orientifold background can reduce the symmetry group $Sl(2,\mathbb{Z})$ to a subgroup Γ_S as discussed at the end of section 2.1.

Let us now check how the Kähler potential and Kähler coordinates transforms under modular transformations (2.21) in Γ_S. We concentrate in the following on the large volume compactification characterized by the α' corrected pre-potential (2.12). Using the explicit expressions (2.15) and (2.16) for G^a, T_α we note that these $\mathcal{N}=1$ coordinates transform under (2.21) as[10]

$$G^a \;\rightarrow\; \frac{G^a}{c\tau+d}\,, \qquad\qquad T_\alpha \;\rightarrow\; T_\alpha + \frac{1}{2}\frac{c\,\mathcal{K}_{\alpha ab}G^a G^b}{c\tau+d}\;. \tag{2.22}$$

where a,b,c,d are the entries of an element of Γ_S. We next analyze how the perturbatively corrected Kähler potential (2.19) transforms under (2.22). It is very easy to evaluate the transformation properties of the first term in (2.19) since

$$(\tau-\bar\tau)^{-1} \quad \rightarrow \quad |c\tau+d|^2(\tau-\bar\tau)^{-1}\;. \tag{2.23}$$

We thus have to focus on the transformation of the combination

$$V_E(\tau,G,T) + \frac{1}{(2i)^{3/2}}\big(\tau-\bar\tau\big)^{3/2}\big[2\zeta(3)\chi - 4\mathrm{Im}\mathcal{F}_{\mathrm{b}}(\tau,G)\big]\;. \tag{2.24}$$

Clearly, the Einstein-frame volume V_E is invariant under Γ_S, since it is a purely geometric quantity. Note however, that invariance does not hold for the α' correction in (2.24). This can be traced back to the fact that we did not include all corrections relevant in

[9] Here we have been a bit sloppy with factors of 2π, which however can be restored easily.

[10] For T_α to transform as in (2.22) we have used that $e^{-\phi/2}v^\alpha$ and $\tilde\rho_\alpha$ are invariant under (2.21). The combination $e^{-\phi/2}v^\alpha$ is precisely the invariant Einstein frame Kähler structure deformation, while $\tilde\rho_\alpha$ arises in the expansion of an $Sl(2,\mathbb{Z})$ invariant $\tilde C_4$ with field strength $F_5 = d\tilde C_4 - \frac{1}{2}dB_2\wedge C_2 + \frac{1}{2}B_2\wedge dC_2$. We have also used that $(\tau-\bar\tau)^{-1} \rightarrow (c\tau+d)^2(\tau-\bar\tau)^{-1} - c(c\tau+d)$.

this large volume limit. Analogously to the discussion in refs. [15, 16] one can argue that also corrections due to D(-1) branes as well as the reduction of D1 instantons have to be included. These couple to the complex dilaton τ and G^a and can complete the α' correction in (2.24) in a modular invariant form. We propose that by including these contributions the large volume Kähler potential $K_{\rm q}$ takes the form

$$K_q = -\ln\big[-i(\tau-\bar\tau)\big] - 2\ln\left[V_E + \frac{1}{2}\chi\, f(\tau,\bar\tau) - 4g(\tau,\bar\tau,G,\bar G)\right], \qquad (2.25)$$

and transforms under Γ_S as

$$e^K \quad\to\quad |c\tau+d|^2 e^K\ . \qquad (2.26)$$

In general it is hard to determine the precise form of the modular invariant forms $f(\tau,\bar\tau)$ and $g(\tau,\bar\tau,G,\bar G)$. In the remainder of this section we will discuss some properties of g, f as well as some candidate modular completions. A calculation of f, g might be possible by restricting the class of Calabi-Yau manifolds to $K3$ fibrations where heterotic-F-theroy duality can be applied.

In the following we will first discuss the modular invariant function $f(\tau,\bar\tau)$ in (2.19). In order to do that, we recall that in ref. [15] a similar problem arose in the computation of the R^4 correction to the ten-dimensional type IIB supergravity action. In this ten-dimensional setup, an additional analysis of the properties of the τ-dependent coefficient $\hat f(\tau,\bar\tau)$ led to the identification

$$\hat f(\tau,\bar\tau) = \sum_{(n,m)\in P} \frac{(\tau-\bar\tau)^{3/2}}{(2i)^{3/2}|m+n\tau|^3}\,, \qquad (2.27)$$

where $P=\mathbb{Z}^2/(0,0)$ is a two-dimensional lattice without the origin. This non-holomorphic Eisenstein series includes indeed the perturbative correction in (2.24), when $n=0$ in the sum (2.27). Moreover, it is invariant under the full group $Sl(2,\mathbb{Z})$ and hence a candidate modular completion of the Kähler potential. It was also conjectured in ref. [16] that the function (2.27) is the correct modular completion of the analog situation in the underlying $\mathcal{N}=2$ theory. In our setup one might want to restrict the sum in (2.27) only to orbits of the subgroup Γ_S. However, in any case modularity together with the limit $n=0$ alone seems not sufficient to fix the form of $f(\tau,\bar\tau)$ in (2.19). Additional conditions such as the singularity structure or the suppression of further mixed contribution are needed to determine $f(\tau,\bar\tau)$ unambiguously. This is in general hard and beyond the scope of this paper. For the general discussion of the superpotential we will simply assume that such a modular completion exists, while for our explicit example in section 3 we will find that $\chi=0$.

Let us also briefly discuss the modular completion $g(\tau,\bar\tau,G,\bar G)$ of the non-perturbative α' corrections inherited from $\mathcal{N}=2$. The corrections we are missing in our computation are the D1 branes dual to the world-sheets inducing the contribution $\mathcal{F}_{\rm b}$. More precisely, we need to include the whole set of (p,q) strings [49, 50] to restore Γ_S duality. Again we are facing the problem that such corrections are hard to compute in general and we can only discuss some candidate solution for g. In ref. [16] the modular completion of the underlying $\mathcal{N}=2$ quaternionic geometry was conjectured to arise from a summation over all $Sl(2,\mathbb{Z})$

images of the world-sheet instanton corrections. In the orientifold limit this leads to the following definition of a modular invariant $\hat{g}$

$$\hat{g}(\tau,\bar{\tau},G,\bar{G}) = \sum_{\beta} n_{k_a} \sum_{(m,n)\in P} \frac{(\tau-\bar{\tau})^{3/2}}{(2i)^{3/2}|n+m\tau|^3} \cos\left((n+\tau m)\frac{k_a(G^a-\bar{G}^a)}{\tau-\bar{\tau}} - mk_aG^a\right). \tag{2.28}$$

This sum encodes all images under $Sl(2,\mathbb{Z})$ of the world-sheet instanton corrections in $\mathrm{Im}\mathcal{F}_{\mathrm{b}}$ divided by stabilizer group generated by shifts $\tau \to \tau+1$. In general one might also want to restrict to orbits of the subgroup Γ_S. It is not hard to check that $\hat{g}$ contains the contribution $\mathrm{Im}\mathcal{F}_{\mathrm{b}}$ for $m=0$. Once again we have to remark that even though $\hat{g}$ has the desired properties, the true correction g is expected to be more complicated. It would thus be desirable to find independent ways to calculate g for specific setups. In the example of section 3 all non-perturbative α' corrections will be absent such that no g is inherited from $\mathcal{N}=2$.

Before moving on to the discussion of the superpotential, let us compare the question of determining $f(\tau,\bar{\tau})$ and $g(\tau,\bar{\tau},G,\bar{G})$ to a somewhat similar situation within topological string theory on a Calabi-Yau threefold [17−20]. The symmetry group in this case is the target space duality group arising from the monodromies around singularities in the moduli space. One can thus attempt to parametrize the non-perturbative corrections by modular forms of this duality group which form a finite ring. Fortunately, the singularity structure for the topological string partition function is often known and additional boundary conditions allow to fix the precise modular forms encoding the non-perturbative corrections at least up to a certain genus. These boundary conditions arise from the singularities of the moduli space or through the application of string-string dualities (see e.g. [19, 20]). One might thus hope that to redo a similar analysis in the $\mathcal{N}=1$ theories discussed in this work. Clearly, one of the obstacles is the non-holomorphicity of the Kähler potential as well as the presence of additional perturbative corrections. For the holomorphic $\mathcal{N}=1$ superpotential this situation is improved as we will discuss in the next section.

2.4 D-instanton superpotentials in type IIB orientifolds

Let us now discuss the D-instanton superpotential arising in type IIB orientifolds with O3/O7 planes. The instantons contributing to the superpotential are typically Euclidean D3 branes wrapped around special four-cycles inside the Calabi-Yau orientifold. In order to give the precise conditions when such a potential arises, one has to embed this orientifold setup into an F-theroy compactification. These conditions have been investigated first by Witten in [23] and later refined for compactifications with background fluxes [51]. Since here our primary interest is the definition of a symmetry invariant superpotential for a generic orientifold compactification, we will directly go to the orientifold and assume that these conditions are satisfied for the cycles under consideration.

In the type IIB orientifolds discussed in the previous sections the instanton superpotential arises from specific Euclidean D3 branes. Let us consider such a brane warp around a devisor Σ in Y/σ. We will pick the devisor such that it non-trivially contributes to the

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superpotential. Schematically these contributions are of the form

$$f(X^I)\ e^{-V_\Sigma + i\phi_\Sigma} \tag{2.29}$$

where V_Σ is the Einstein-frame volume of Σ and ϕ_Σ is the integral of the R-R four-form C_4 over Σ. The function $f(X^I)$ can depend on other chiral multiplets in the spectrum and we will be the main focus of our considerations. Before turning to the discussion of f, let us first note that the form of the exponential is not yet exact, since we are missing the coupling to the lower R-R forms and the B-field in the exponential. Recall that the effective action on the word-volume of the Euclidean D3 brane takes the form

$$S^{D3} = iT_{D3}\int_{\mathcal{W}_4} d^4\lambda e^{-\phi}\sqrt{\det\left(g - B_2 + F\right)} + T_{D3}\int_{\mathcal{W}_4} C^{\rm RR}\wedge e^{-B_2+F}\ , \tag{2.30}$$

where $C^{\rm RR} = C_0 + C_2 + C_4$ are the Ramond-Ramond fields and F is the fieldstrength on the brane. The first and second term correspond to the Born-Infeld and Chern-Simons coupling respectively. In order that the D-instanton preserves supersymmetry it has to wrap a supersymmetric cycle. Applying the standard calibration conditions for supersymmetric branes we find that the correct couplings to the R-R forms and the B-field [52]. The correct superpotential contribution is thus proportional to

$$\exp\Big[-\frac{1}{2}\int_\Sigma e^{-\phi}\big(J\wedge J - B_2\wedge B_2\big) - i\int_\Sigma\big(C_4 - C_2\wedge B_2 + \tfrac{1}{2}C_0 B_2\wedge B_2\big)\Big]\ . \tag{2.31}$$

Note that the first term under the first integral is V_Σ, since the Kähler form J is evaluated in the string-frame metric. The expression (2.31) is precisely $\exp(-i\int\rho_c)$ with ρ_c introduced in (2.8). Thus we find that the generic superpotential is of the expected form

$$W_{\text{D-inst}} = \sum_\Sigma f_\Sigma(X^I)\ e^{in_\Sigma{}^\alpha T_\alpha}\ , \qquad\qquad n_\Sigma^\alpha = \int_\Sigma \tilde\omega^\alpha\ , \tag{2.32}$$

where n_Σ^α are integers for $\Sigma\in H_4(Y,\mathbb{Z})$ and $\tilde\omega^\alpha\in H^4_+(Y,\mathbb{Z})$. We are now in the position to discuss the moduli dependence of $f(X)$ in more detail.

So far we did not discuss the holomorphic function $f(X)$. In general, it can depend on various other moduli $\{X^I\}$ of the orientifold or underlying F-theory compactification. As in (2.11) we denote the elliptically fibered fourfold corresponding to the orientifold by Y_4. The moduli dependence of f can arise from:

(a) the complex structure deformations of Y_4: in the orientifold limit these include the complex dilaton τ corresponding to the complex structure of the elliptic fiber, the complex structure deformations of Y/σ as well as the D7 brane moduli,

(b) the $h^{(2,1)}$ complex scalars arising in the expansion of C_M in $H^{(2,1)}(Y_4)$: these include the complex scalars G^a as well as Wilson lines of the D7 brane,

(c) the complex coordinates x^i labeling the position of space-time filling D3-branes in Y_4 or Y/σ.

In the following we will discuss $f(X)$ as a function of the complex dilaton τ, the moduli G^a arising by expanding the type IIB NS-NS and R-R two-form. An analysis of the dependence of $f(X)$ on the positions of the space-time filling D3-branes x^i on Y/σ can be found in [27, 53].

It turns out that a direct computation of the function $f(X)$ is in general very hard and involves the evaluation of appropriate determinants [23]. However, we can already learn much about f by studying the transformation properties of the superpotential and the Kähler potential under shifts and modular transformations. This was already initiated in refs. [26, 27] for M- and F-theory compactifications were only a local analysis can be performed. Here we will make this discussion very concrete for the type IIB orientifolds studied in section 2.1 and focus on its dependence on τ, G^a decomposing $f(X) = A_0 \Theta(\tau, G^a)$, with A_0 depending on the remaining moduli. We thus write

$$W_{\text{D-inst}} = A_0 \sum_{\Sigma} \Theta_{\Sigma}(\tau, G) e^{i n_{\Sigma}{}^{\alpha} T_{\alpha}} \ . \tag{2.33}$$

Let us now investigate the transformation properties of the coefficients $\Theta_\Sigma(\tau, G^a)$ in more detail. We will first discuss the duality transformations induced by modular changes of the complex dilaton τ as given in (2.21). In section 2.3 we have argued that e^K transforms as given in equation (2.26) under modular transformations. From this we conclude that the superpotential has to change as[11]

$$W \quad \rightarrow \quad (c\tau + d)^{-1} W \ . \tag{2.34}$$

To see this, we note that the combination $e^K |W|^2$ has to be invariant since it determines, for example, in the physical gravitino mass. Equation (2.34) exactly states that W has to be a modular form of weight -1 under the duality group Γ_S. Let us note that this is obviously true for the flux superpotential $W = \int \Omega \wedge (F_3 - \tau H_3)$ in (2.10). For the D-instanton superpotential (2.33) we will see momentarily, that this imposes constraints on the functions $\Theta_\Sigma(\tau, G)$.

The second transformation we will consider are the shifts (2.20) in the NS-NS two-form B_2. More precisely, let us transform the orientifold coordinates by $b^a \rightarrow b^a + 2\pi n^a$. From the definitions (2.15) and (2.16) of the coordinates G^a, T_α we deduce that

$$\begin{aligned} G^a &\quad \rightarrow \quad G^a - 2\pi\tau n^a \ , \\ T_\alpha &\quad \rightarrow \quad T_\alpha - 2\pi \mathcal{K}_{\alpha ab} n^a G^b + 2\pi^2 \tau \mathcal{K}_{\alpha ab} n^a n^b \ . \end{aligned} \tag{2.35}$$

As we have seen in section 2.3, it is not hard to check that this is a symmetry of the orientifold Kähler potential. Due to the invariance of the combination $e^K |W|^2$ we conclude that W can only transform by a trivial phase factor and is otherwise invariant. Invariance of W together with the fact that T_α transforms as in (2.35) restricts the coefficient functions $\Theta_\Sigma(\tau, G)$ of the instanton superpotential (2.33) as we will discuss next.

[11] In the following we will not include a possible phase. For a related discussion of the possibility to include such a phase factor see, for example, refs. [13, 14].

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We can now infer the properties of the functions $\Theta_\Sigma(\tau, G)$ appearing in (2.33). Our strategy is to use the fact that W is a modular form of weight -1 but otherwise invariant under (2.21), (2.22) and (2.35). Since $e^{in_\Sigma^\alpha T_\alpha}$ in (2.33) transforms non-trivially under these symmetries also $\Theta_\Sigma(\tau, G)$ has to transform in order to ensure the correct modular properties of W. It turns out that the Θ's are generalizations of the well-known theta functions, or more precisely appropriate holomorphic Jacobi forms.[12] To summarize their properties we simplify our analysis and restrict our attention to the case where only one $T \equiv T_{\alpha'}$ transforms non-trivially under the above groups. In other words, we will assume here that the only non-vanishing intersection with negative indices is $\mathcal{K}_{\alpha' ab} = -C_{ab}$. We also denote $n_\Sigma^{\alpha'} = n$. The Jacobi form $\Theta_n(\tau, G)$ then turns out to be of weight -1 and index n. In other words, under the transformation (2.22) this form transforms as

$$\Theta_n(\tau, G) \;\rightarrow\; (c\tau + d)^{-1} \exp\!\left(\frac{ni}{2}\frac{c\, C_{ab}G^aG^b}{c\tau + d}\right) \Theta_n(\tau, G)\,, \tag{2.36}$$

which is consistent with the required transformation behavior (2.34). Also the transformation (2.35) of e^{inT} is cancelled by the corresponding Jacobi form Θ_n since

$$\Theta_n(\tau, G) \;\rightarrow\; \exp\big(-2\pi i n C_{ab} n^a G^b + 2\pi^2 i n \tau C_{ab} n^a n^b\big)\; \Theta_n(\tau, G) \tag{2.37}$$

under the transformation (2.35). Carefully restoring factors of 2π the transformations (2.36) and (2.37) are exactly the transformation properties of Jacobi forms. For only one field G^a, the theory of Jacobi forms is extensively reviewed by Eichler and Zagier in ref. [54]. The more general situation including vectors G^a is discussed, for example, in the work of Borcherds [55] (section 3).

Before turning to the example in the next section, let us summarize some classical results about candidate Jacobi forms Θ_n [54, 55]. In order to do that we introduce the theta functions of weight $s/2$ and index m by setting

$$\theta_{(m)\ L+r}(\tau, G) = \sum_{n_a \in L+r} e^{i\tau n^2/2} e^{miG^a n_a}\,, \qquad n^2 = C^{ab} n_a n_b\,, \tag{2.38}$$

where L is some positive definite rational lattice of dimension s, and r is some vector which admits an expansion in a basis of L with rational coefficients. It can be shown that any Jacobi form Θ_n can be written as a sum of products of the theta functions $\theta_{(m)\ L+r}$ and modular forms $\tilde{\eta}(\tau)$. Heuristically, we can write

$$\Theta_n(\tau, G) = \sum \frac{\theta_{(n)}(\tau, G)}{\tilde{\eta}(\tau)}\ . \tag{2.39}$$

This form is well known from various other perspectives. For example, it was shown in [56] that the partition function of a chiral boson on a genus one surface is of this form. More importantly, also the partition function of the M5 brane takes a form similar to (2.39) as was first discussed in ref. [26]. This is no surprise, since we know that the F-theory lift of

[12]Holomorphicity here only means that $\Theta_\Sigma(\tau, G)$ is independent of $\bar{\tau}, \bar{G}^a$ and does not restrict the singularity structure.

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the D3 instantons are six-dimensional branes. Analyzing F-theory from the M-theory point of view as mentioned at the end of section 2.1 these six-dimensional branes are M5 branes wrapped around four-cycles in the base B_3 of (2.11) as well as on the two-dimensional fiber.

Clearly, an important task is to explicitly find the correct Jacobi forms $\Theta_n(\tau, G)$ for specific examples. One suspects that this problem is more tractable then determining the modular corrections to the Kähler potential due to the holomorphicity of W and the absence of perturbative corrections. Ideally, one likes to use physical arguments, for example on the singularity structure of W, to restrict the set of candidate Jacobi forms to a finite set. Computing W in a particular limit, e.g. an orbifold limit, might then determine the correct linear combination to appear in the full W. In the next section, we will take a different route in the study of the Enriques orientifold. We will use some intuition from the topological strings on the Enriques Calabi-Yau to propose a candidate W including non-trivial Jacobi forms Θ_n.

3. D-instantons and the Enriques orientifold

In this section we discuss one type IIB orientifold compactification in more detail and illustrate some of the general story outlined in the previous section. We construct an orientifold of the Enriques Calabi-Yau Y_E and argue that the quantum corrections are under particular control. It is also shown how the $\mathcal{N} = 1$ Kähler manifold $\tilde{\mathcal{M}}_{\mathrm{q}}$ inside the $\mathcal{N} = 2$ quaternionic space can be identified with the original special Kähler moduli space times a $Sl(2, \mathbb{R})/\,\mathrm{U}(1)$ factor. In this duality the new complex coordinates contain the R-R fields as in (2.8) and provide the correct couplings to D-instantons. We use this identification to translate instanton expansions known from topological string theory on Y_E to the corresponding physical orientifold setup. This leads us to propose a specific D-instanton superpotential for the Enriques orientifold.

3.1 Enriques Calabi-Yau and counting of D(-1)-D1-D3 states

Let us begin by reviewing some basic facts about the Enriques Calabi-Yau Y_E and its moduli space. The Enriques Calabi-Yau takes the form $Y_E = (K3 \times \mathbb{T}^2)/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ acts as an inversion of the complex coordinate of $\mathbb{T}^2$ and as the Enriques involution on $K3$ [28−30]. Y_E has holonomy group $\mathrm{SU}(2) \times \mathbb{Z}_2$. This implies that type II string theory compactified on the Enriques Calabi-Yau will lead to a four-dimensional theory with $\mathcal{N} = 2$ supersymmetry. Nevertheless, due to the fact that it does not have the full SU(3) holonomy of generic Calabi-Yau threefolds, various special properties of $\mathcal{N} = 4$ compactifications on $K3 \times \mathbb{T}^2$ are inherited.

In order to discuss the moduli space of Y_E we first need to summarize the cohomology on this Calabi-Yau manifold. We review in appendix A that the two-form and three-from integral cohomologies can be identified with the following lattices [29]

$$H^2(Y_E, \mathbb{Z}) \cong \mathbb{Z} \oplus \Gamma^{1,1} \oplus \Gamma_{E_8}(-1)\,, \tag{3.1}$$

$$H^3(Y_E, \mathbb{Z}) \cong \big(\Gamma^{1,1} \oplus \Gamma_{E_8}(-1) \oplus \Gamma^{1,1}_g\big) \oplus \big(\Gamma^{1,1} \oplus \Gamma_{E_8}(-1) \oplus \Gamma^{1,1}_g\big)\,, \tag{3.2}$$

where $\Gamma^{1,1}$ is a two-dimensional lattice with signature $(1,1)$ and inner product $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $\Gamma_{E_8}(-1)$ has an inner product given by -1 times the Cartan matrix of the exceptional group E_8. We denote an integral basis $(\omega_A) = (\omega_S, \omega_i, \omega_a)$ of $H^2(Y_E, \mathbb{Z})$, where ω_S, ω_i and ω_a are basis elements of the three terms in (3.1) respectively. We already defined the triple intersections $\mathcal{K}_{ABC}$ in (2.13). Using the relation to the underlying $K3 \times \mathbb{T}^2$ one shows that the only non-vanishing intersections are

$$\mathcal{K}_{S12} = \mathcal{K}_{S21} = 1\,, \qquad \mathcal{K}_{Sab} = -C_{ab}\,, \tag{3.3}$$

where in the appropriate basis the inverse C^{ab} of C_{ab} is the Cartan matrix of E_8 as already mentioned before. As in section 2.1 we also introduce a basis $(\tilde{\omega}^A) = (\tilde{\omega}^S, \tilde{\omega}^i, \tilde{\omega}^a)$ of $H^4(Y_E, \mathbb{Z})$ dual to ω_A. Finally, we will need to introduce a real symplectic basis (α_A, β^A) of the third cohomology $H^3(Y_E, \mathbb{Z})$.

The explicit form (3.1) and (3.2) of the integral cohomology of Y_E allows us to read of the dimensions $h^{(p,q)}$ of the cohomologies $H^{(p,q)}(Y_E)$. We find that

$$h^{(1,1)}(Y_E) = h^{(2,1)}(Y_E) = 11\ . \tag{3.4}$$

This implies that the moduli spaces of complex structure deformations $\mathcal{M}_{\text{cs}}$ as well as of Kähler structure deformations $\mathcal{M}_{\text{ks}}$ are both complex eleven-dimensional. Moreover, one shows that both of these spaces are the coset [29]

$$\mathcal{M}_{\text{cs/ks}} \;=\; Sl(2,\mathbb{R})/\,\mathrm{U}(1) \;\times\; O(10,2)/\big(O(10)\times O(2)\big)\,, \tag{3.5}$$

where $O(q,p,\mathbb{R})$ are orthogonal groups with values in the real numbers. The identification $\mathcal{M}_{\text{cs}} \cong \mathcal{M}_{ks}$ arises due to the fact that the Enriques Calabi-Yau is self-mirror. In a careful treatment one also finds that these cosets have to be divided by the discrete symmetry group

$$O_E(\mathbb{Z}) \;\equiv\; Sl(2,\mathbb{Z}) \times O(10,2,\mathbb{Z})\,, \tag{3.6}$$

which is a non-perturbative symmetry of string theory on Y_E. The presence of this discrete factor is of central importance. All functions on $\mathcal{M}_{\text{cs/ks}}$ have to transform covariantly under $O_E(\mathbb{Z})$ to be well defined. Furthermore, note that after dividing by $O_E(\mathbb{Z})$ the identification (3.5) is exact and receives no corrections due to world-sheet instantons [29, 57]. As we will discuss next this implies that the Enriques Calabi-Yau is a special example with an exact pre-potential cubic in the moduli around the large volume or large complex structure point. To make this more precise we discuss the geometry of the moduli space $\mathcal{M}_{\text{ks}}$ in more detail. Clearly, due to the fact that Y_E is self-mirror the geometry of $\mathcal{M}_{\text{cs}}$ takes a similar form.

Compactifying Type II string theory on the Enriques Calabi-Yau yields an effective four-dimensional theory with $\mathcal{N}=2$ supersymmetry. In general, the $\mathcal{N}=2$ scalar moduli space consists of a special Kähler $\mathcal{M}_{\text{sk}}$ times a quaternionic manifold $\mathcal{M}_{\text{q}}$. For the Enriques Calabi-Yau both spaces are cosets. Since we are interested in type IIB compactifications we find that the complex structure deformations are the space $\mathcal{M}_{\text{sk}}$ while the Kähler structure deformations sit inside the quaternionic space $\mathcal{M}_{\text{q}}$. One finds [29]

$$\mathcal{M}_{\text{sk}} = \mathcal{M}_{\text{cs}}\,, \qquad\qquad \mathcal{M}_{\text{q}} = O(12,4)/\big(O(12)\times O(4)\big) \;\supset\; \mathcal{M}_{\text{ks}}\ . \tag{3.7}$$

Note that $\mathcal{M}_{\rm sk}$ is exact and receives no perturbative corrections or corrections due to world-sheet or D-instantons. In contrast, $\mathcal{M}_{\rm q}$ is in general perturbatively and non-perturbatively corrected. The geometry of the two moduli spaces in (3.7) is encoded by two cubic pre-potentials. For $\mathcal{M}_{\rm sk}$ one finds around the large complex structure point a pre-potential of the form[13]

$$\tilde{\mathcal{F}}(z) = -z^S z^1 z^2 + \frac{1}{2} z^S C_{ab} z^a z^b \ . \tag{3.8}$$

Due to the absence of world-sheet instanton corrections this potential is exact and can be transformed and used at other points in the moduli space $\mathcal{M}_{\rm sk}$. This special Kähler manifold encodes deformations of the complex structure through the holomorphic $(3,0)$ form

$$\Omega(z) = X^K(z)\alpha_K - \tilde{\mathcal{F}}_K(z)\beta^K \ , \tag{3.9}$$

where (α_K, β^K) is a real symplectic basis of $H^3(Y_E, \mathbb{Z})$. The periods of Ω are thus $(X^K, \tilde{\mathcal{F}}_K)$, where $\tilde{\mathcal{F}}_K$ is the derivative of $\tilde{\mathcal{F}}(z)$ with respect to X^K. In the spacial coordinates z above one has $z^S = X^S/X^0$, $z^i = X^i/X^0$ and $z^a = X^a/X^0$. One can thus rewrite $\tilde{\mathcal{F}}_K$ as derivatives with respect to the coordinates z [58].

The quaternionic manifold $\mathcal{M}_{\rm q}$ can be constructed by starting with the underlying special Kähler manifold $\mathcal{M}_{\rm ks}(t)$. The coordinates $t^A = (S, t^i, t^a)$ are the complexified Kähler structure deformations of Y_E arising in the expansion of $-B_2 + iJ$ into the two-form basis $\omega_A = (\omega_S, \omega_i, \omega_a)$. The geometry of the special Kähler manifold is determined by the pre-potential[14]

$$\mathcal{F}(t) = -St^1t^2 + \frac{1}{2!} SC_{ab}t^a t^b \ . \tag{3.10}$$

It is straightforward to derive the corresponding Kähler potential $K_{\rm ks}(S, \bar{S}, t, \bar{t})$. In general, $K_{\rm ks}$ can be obtained from the even form ρ introduced in (2.7) by setting $K_{\rm ks} = -\ln i\langle \rho, \bar{\rho}\rangle$ with wedge product defined in footnote 4. Inserting (3.10) into this expression one evaluates

$$K_{\rm ks} = -\ln\big(i(S-\bar{S})Y\big) \ , \qquad Y = (t-\bar{t})^1(t-\bar{t})^2 - \tfrac{1}{2}(t-\bar{t})^a(t-\bar{t})^b C_{ab} \ . \tag{3.11}$$

The classical quaternionic geometry can be obtained from $\mathcal{M}_{\rm ks}$ by applying the c-map construction [59]. Since our focus will be the orientifold scenario, we will not review the details here. Let us however note that the quaternionic geometry is invariant under the Kähler transformations of $K_{\rm ks}$. It is therefore naturally formulated in terms of the invariant combination $C\rho$, with C proportional to the dilaton $e^{-\phi}$. Note that C and ρ itself do transform under the Kähler transformations $K_{\rm ks} \rightarrow K_{\rm ks} - f(t) - \bar{f}(\bar{t})$ as

$$C \quad \rightarrow \quad e^{-f}\, C \ , \qquad\qquad \rho \quad \rightarrow \quad e^{f}\, \rho \ , \tag{3.12}$$

where $f(t)$ is a holomorphic function of the moduli.

We will now go one step further and discuss a first set of quantum corrections depending on the moduli of $\mathcal{M}_{\rm ks}$. Following [35, 36, 31] we will introduce a functional $\Phi_{\rm B}$ which counts

[13] A more careful analysis reveals that there is a linear term $-z^S$ in $\tilde{\mathcal{F}}(z)$ [31]. This term however does not appear in the Kähler potential and hence not in any physical object discussed in the following.

[14] As in (3.8) we ignore a linear term in S which can be absorbed into a redefinition of the coordinates on $\mathcal{M}_{\rm q}$.

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the leading degeneracies of D(-1), D1, D3 states on the Enriques fiber. Before recalling the precise form of these corrections let us note that this investigation will not take place in the large volume limit but rather at a second special locus of the Enriques moduli space. At this locus also Euclidean D3 branes wrapped around a the Enriques fiber are becoming light. To make this more precise, we will choose 'dual' coordinate $\mathcal{T}^1, \mathcal{T}^2, \mathcal{T}^a$ in which large $\mathrm{Im}\mathcal{T}$ implies a small volume of the $K3$. The transformations from the large volume limit to this special Enriques locus is given by

$$\mathcal{T}^2 = -\frac{1}{2t^2}\,, \qquad \mathcal{T}^1 = \frac{1}{t^2}\left(t^1 t^2 - \frac{1}{2}C_{ab}t^a t^b\right)\,, \qquad \mathcal{T}^a = -\frac{1}{t^2}t^a\ . \tag{3.13}$$

Under this change of coordinates we find that Y defined in (3.11) transforms as

$$2Y = \frac{1}{2\,\mathcal{T}^2\bar{\mathcal{T}}^2}\left[2(\mathcal{T}-\bar{\mathcal{T}})^1(\mathcal{T}-\bar{\mathcal{T}})^2 - (\mathcal{T}-\bar{\mathcal{T}})^a(\mathcal{T}-\bar{\mathcal{T}})^b C^D_{ab}\right] = \frac{1}{\mathcal{T}^2\bar{\mathcal{T}}^2}Y_D \tag{3.14}$$

where we have introduced $C^D_{ij} = C_{ij}$, $C^D_{ab} = \frac{1}{2}C_{ab}$ and defined Y_D. In other words, defining the dual Kähler potential $K_D(S,\mathcal{T})$ as

$$K_D(S,\mathcal{T}) = -\ln\left(i(S-\bar{S})Y_D\right)\,, \tag{3.15}$$

one finds that K_{ks} and K_D differ only by a Kähler transformation.[15] From the coordinate definition (3.13) one concludes that the corresponding cohomology lattice is

$$\Gamma^{1,1}_s \oplus \Gamma_{E_8}(-2) \cong H^0(E,\mathbb{Z}) \oplus H^4(E,\mathbb{Z}) \oplus \Gamma_{E_8}(-2) \tag{3.16}$$

where $H^0(E,\mathbb{Z})$ and $H^4(E,\mathbb{Z})$ are the zero and four cohomology of the Enriques fiber. This can be seen as follows. The Kähler invariant combination to consider is $C\rho$ with C and ρ transforming as in (3.12). One can thus remove the overall factor of $1/t^2$ in the definitions (3.13). On the one hand this leads to $\mathcal{T}^2 \propto C$ such that $\mathcal{T}^2$ scales the element in $H^0(E)$. On the other hand $\mathcal{T}^1 \propto C(2t^1t^2 - C_{ab}t^at^b)$ which is the square of the complexified Kähler form and hence parametrizes $H^4(E)$. We also see that the lattice (3.16) contains the self-dual lattice $\Gamma_{E_8}(-2)$ which has intersection form $C^{D\,ab} = 2C^{ab}$. The extra factor 2 arises due to the factor 1/2 in the definition of $\mathcal{T}^2$. We will see in the next section that the coordinates $\mathcal{T}^1, \mathcal{T}^2, \mathcal{T}^a$ have a second advantage, since they can be identified with the $\mathcal{N}=1$ coordinates of the orientifold theory.

We are now in the position to recall a functional $\Phi_{\text{B}}(\mathcal{T})$ counting the leading degeneracies of Euclidean D(-1), D1, D3 branes on the Enriques fiber. It was shown in refs. [33, 34], that for $\mathcal{T}^i, \mathcal{T}^a$ with $Y_D < -1$ one defines a convergent functional

$$\Phi_{\text{B}}(\mathcal{T}) = e^{i\mathcal{T}^1}\prod_{r\in\Pi^+}(1-e^{ir\cdot\mathcal{T}})^{(-1)^{m+n}c_{\text{B}}(r^2/2)}\,, \tag{3.17}$$

where $r\cdot\mathcal{T} = n\mathcal{T}^1 + m\mathcal{T}^2 - C^D_{ab}r^a\mathcal{T}^b$ for vectors $r = (m,n,r^a)$ in the lattice (3.16). In the product (3.17) we denote by Π^+ the set of positive roots of the fake monster Lie

[15] One finds that $K_{\text{ks}} = K_D - f - \bar{f}$, where $f = -\ln\left(i\sqrt{2}\,\mathcal{T}^2\right)$.

superalgebra consisting of all nonzero vectors r with $r^2 = 2mn - C^D_{ab} r^a r^b \geq -2$ such that $m > 0$, or $m = 0$ and $n > 0$. The exponents $c_{\rm B}(r^2/2)$ are given via the modular form

$$\sum_n c_{\rm B}(n) q^n = \frac{\eta(q^2)^8}{\eta(q)^8 \eta(q^4)^8} , \qquad\qquad r^2/2 = n , \tag{3.18}$$

where $\eta(q)$ is the standard eta function. It was argued in ref. [31] that $\Phi_{\rm B}(\mathcal{T})$ counts the degeneracies of D(-1), D1, D3 branes on the Enriques fiber. To show this Klemm and Mariño [31] applied a similar argument as Gopakumar and Vafa [47] by performing a Schwinger calculation including the light states at the moduli space locus parametrized by $\mathcal{T}^i, \mathcal{T}^a$. The corresponding BPS particles are bound states of D3 branes wrapping the Enriques fiber, D1 wrapped around the curves in the E_8 sublattice in (3.16) and D(-1) branes. The leading degeneracies are counted by the lowest genus free-energies $\mathcal{F}^{(g)}$ of the topological string on Y_E. Since $\mathcal{F}^{(0)}$ is trivial for the Enriques Calabi-Yau the first non-trivial contribution arises from a resummation of $\mathcal{F}^{(1)}$ which precisely contains the holomorphic function $\Phi_{\rm B}(\mathcal{T})$. It is important to remark, that $\Phi_{\rm B}(\mathcal{T})$ has particularly nice modular properties as we will discuss in section 3.3. For contributions from the higher $\mathcal{F}^{(g)}$ this is only the case if also a non-holomorphic dependence is included. Therefore, we will propose in section 3.3 that $\Phi_{\rm B}$ might contain the leading contribution to a holomorphic and modular superpotential of the orientifold theory on the Enriques Calabi-Yau.

3.2 Effective action for the Enriques orientifold

In this section we study the effective four-dimensional $\mathcal{N} = 1$ supergravity obtained by compactifying type IIB supergravity on an orientifold of the Enriques Calabi-Yau Y_E. In order to do this we first have to define an involution σ on Y_E and investigate its action on the cohomology. It was shown in refs. [60, 30] that involutions on the Enriques surface can be characterized by their action on the lattice (3.1). In particular, there exist an involution acting with a minus sign on the $\Gamma_{E_8}(-1)$ term in (3.1), while leaving the $\Gamma^{1,1}$ term invariant. We complete this involution by also inverting the $\mathbb{P}^1 \cong T^2/\mathbb{Z}_2$ base of the fibration. This keeps the volume form of $\mathbb{P}^1$ invariant. We thus find for the second cohomology lattice (3.1) the split

$$H^2_+(Y_E, \mathbb{Z}) \;\cong\; \mathbb{Z} \oplus \Gamma^{1,1} , \qquad H^2_-(Y_E, \mathbb{Z}) \;\cong\; \Gamma_{E_8}(-1) , \tag{3.19}$$

where $H^2_\pm$ are the plus and minus eigenspaces of σ^*. An integral basis $\omega_A = (\omega_S, \omega_i, \omega_a)$ of $H^2(Y_E, \mathbb{Z})$ is introduced by setting

$$\omega_\alpha = (\omega_S, \omega_i) \in H^2_+(Y_E, \mathbb{Z}) , \qquad\qquad \omega_a \in H^2_-(Y_E, \mathbb{Z}) \; . \tag{3.20}$$

This is consistent with the basis ω_A introduced in the previous section. The non-vanishing triple intersections $\mathcal{K}_{Sij}$ and $\mathcal{K}_{Sab}$ where already given in (3.3). It is important to note that the orientifold constraints (2.14) are indeed satisfied, since $\mathcal{K}_{abc}$, $\mathcal{K}_{a\alpha\beta}$ vanish for α, β running over S, i.

The odd cohomology $H^3(Y_E, \mathbb{Z})$ also splits into positive and negative eigenspaces under the involution. In order to make this split explicit, we note that the above σ can be extended to the underlying $K3$ surface such that it acts with a minus sign on the $\Gamma_{E_8}(-1)$ terms in the second cohomology lattice $H^2(K3,\mathbb{Z})$ given in (A.1), while keeping the remaining terms invariant. This is of course consistent with the split of the two-cohomology (3.19). The third cohomology $H^3(Y_E,\mathbb{Z})$ of the Enriques Calabi-Yau is obtained by wedging one-forms of the T^2 with two-forms of the $K3$ both anti-invariant under the $\mathbb{Z}_2$ involution defining the Enriques Calabi-Yau. Also including the negative sign of σ on the two one-forms of $T^2/\mathbb{Z}_2$ we thus find that (3.2) splits as

$$\begin{aligned} H^3_+(Y_E,\mathbb{Z}) &\cong \Gamma_{E_8}(-1)\oplus\Gamma_{E_8}(-1)\,, \\ H^3_-(Y_E,\mathbb{Z}) &\cong \left(\Gamma^{1,1}\oplus\Gamma^{1,1}_g\right)\oplus\left(\Gamma^{1,1}\oplus\Gamma^{1,1}_g\right)\,. \end{aligned} \tag{3.21}$$

We are now in the position to discuss the reduction of the moduli spaces following the general approach in section 2.1.

Let us first discuss the reduction of the $\mathcal{N}=2$ special Kähler manifold $\mathcal{M}_{\rm cs}$ spanned by the complex structure deformations $z^\alpha=(z^S,z^i)$ and z^a. From (2.1) we note that the holomorphic three-form Ω is an element of the negative eigenspace of σ^*. This implies that in the orientifold setup we have $z^a=0$ and the expansion (3.9) reduces to

$$\begin{aligned} \Omega &= X^0(\alpha_0+z^\alpha\alpha_\alpha-\tilde{\mathcal{F}}_{z^\alpha}\beta^\alpha-(2\tilde{\mathcal{F}}-z^\alpha\tilde{\mathcal{F}}_{z^\alpha})\beta^0) \\ &= X^0(\alpha_0+z^\alpha\alpha_\alpha+z^1z^2\beta^S+z^Sz^2\beta^1+z^Sz^1\beta^2+z^Sz^1z^2\beta^0)\,, \end{aligned} \tag{3.22}$$

where $(\alpha_0,\alpha_\alpha,\beta^\alpha,\beta^0)$ is a real symplectic basis of $H^3_-(Y_E,\mathbb{Z})$ given in (3.21). The prepotential for this reduced special Kähler manifold $\tilde{\mathcal{M}}_{\rm sk}(z)$ is thus a function of the three moduli $z^\alpha=(z^S,z^i)$ only and takes the form $\tilde{\mathcal{F}}(z^I)=-z^Sz^1z^2$. The Kähler potential is evaluated explicitly to be of the form

$$K_{\rm cs} = -\ln\left[i\int\Omega(z)\wedge\bar{\Omega}(\bar{z})\right] = -\ln\left[i(z^S-\bar{z}^S)(z^1-\bar{z}^1)(z^2-\bar{z}^2)\right]\,, \tag{3.23}$$

where we have removed the fundamental period X^0 by a Kähler transformation. The geometry of this reduced moduli space $\tilde{\mathcal{M}}_{\rm cs}$ has been studied intensively in the literature [31, 20]. It can be shown that the mirror map takes a particularly simple form due to the absence of world-sheet instantons. It respects the discrete target space symmetry $Sl(2,\mathbb{Z})\times\Gamma(2)\times\Gamma(2)$ in the three coordinates z^S, z^i and can be given in terms of modular functions of these groups. Note that in addition to the chiral multiplets just discussed, the projected Enriques theory also admits $h^{(2,1)}_+=8$, $\mathcal{N}=1$ vector multiplets A_a. The gauge-kinetic coupling function has to be holomorphic and is simply given by

$$f_{ab}(z) = -iC_{ab}z^S\,. \tag{3.24}$$

The kinetic term for A_a has coupling matrix $\frac{1}{2}\mathrm{Re}(f_{ab})=\frac{1}{2}C_{ab}\mathrm{Im}z^S$ and is indeed positive definite for $\mathrm{Im}z^S>0$.

Let us now turn to the discussion of the Kähler moduli space $\tilde{\mathcal{M}}_{\rm q}$ inside the quaternionic moduli space $\mathcal{M}_{\rm q}$. In (2.4) and (2.6) we already specified the orientifold invariant expansions of the Kähler form J, the NS-NS two-form B_2 and the R-R forms C_2, C_4. In the basis introduced in (3.20) we can summarize these expansions as

$$J=v^S\omega_S+v^i\omega_i\,,\qquad B_2=b^a\omega_a\,,\qquad C_2=c^a\omega_a\,,\qquad C_4=\rho_S\tilde{\omega}^S+\rho_i\tilde{\omega}^i\,, \tag{3.25}$$

where the basis $(\tilde{\omega}^S, \tilde{\omega}^i)$ of $H^4_+(Y_E, \mathbb{Z})$ is chosen to be dual to (ω_S, ω_i). The real scalar fields v^a, ρ_a as well as b^S, b^i, c^S, c^i have to vanish i.e. are projected out by the orientifold. The $\mathcal{N}=1$ coordinates on the Kähler manifold $\tilde{\mathcal{M}}_{\rm q}$ are obtained by expanding the complex even form ρ_c as in (2.8). This implies that the coordinates τ, G^a are exactly as given in (2.15). The coordinates $T_\alpha = (T_S, T_i)$ take the same form as the large volume result (2.16) due to the absence of world-sheet instantons in the Enriques Calabi-Yau. Explicitly, one evaluates

$$\begin{aligned} T_S &= i e^{-\phi} v^1 v^2 - \tilde{\rho}_S + \frac{1}{2(\tau-\bar{\tau})} C_{ab} G^a (G-\bar{G})^b\,, \\ T_i &= \frac{1}{2} i e^{-\phi} v^S v^j - \rho_i\,, \qquad\qquad i,j = 1,2\,, \quad i \neq j\,, \end{aligned} \tag{3.26}$$

where $\tilde{\rho}_S = \rho_S - \frac{1}{2} C_{ab} c^a b^b$. The $\mathcal{N}=1$ Kähler potential can be also deduced from our general considerations in section 2.1. More precisely, one uses (3.26) together with (2.9) or (2.19) to evaluate

$$\begin{aligned} K_{\rm q} = &-\ln\left[\frac{1}{4} i(T_1-\bar{T}_1)\big(2(T_S-\bar{T}_S)(\tau-\bar{\tau}) - C_{ab}(G-\bar{G})^a(G-\bar{G})^b\right] \\ &-\ln\big[-i(T_2-\bar{T}_2)\big]\,. \end{aligned} \tag{3.27}$$

This simple explicit form of $K_{\rm q}$ arises due to the special form of the intersections (3.3) and the simple cubic pre-potential (3.10). Note that $K_{\rm q}$ is not corrected by $\mathcal{N}=2$ α' contributions, since these vanish identically for the Enriques Calabi-Yau. In particular, one notices that the perturbative α' corrections proportional to the Euler characteristic $\chi(Y_E)$ vanish due to $\chi(Y_E) = 2(h^{(1,1)} - h^{(2,1)}) = 0$. We thus conclude that the $\mathcal{N}=1$ Enriques orientifold theory is particularly well under control due to the simplicity of the underlying $\mathcal{N}=2$ theory. The $\mathcal{N}=1$ moduli space $\tilde{\mathcal{M}}_{\rm q}$ is also a coset, which is evaluated to be of the form

$$\tilde{\mathcal{M}}_{\rm q} \;=\; Sl(2,\mathbb{R})/\,\mathrm{U}(1) \;\times\; \Big(Sl(2,\mathbb{R})/\,\mathrm{U}(1) \;\times\; O(10,2)/\big(O(10)\times O(2)\big)\Big)\,. \tag{3.28}$$

Remarkably, we find that the original $\mathcal{N}=2$ special Kähler manifold $\mathcal{M}_{\rm ks}$ given in (3.5) arises as the second factor of $\tilde{\mathcal{M}}_{\rm q}$. Such a phenomenon was already studied from a supergravity point of view in refs. [61]. In the following we will discuss this duality in more detail and make contact to the second parametrization of $\mathcal{M}_{\rm ks}$ introduced in (3.13).

Let us now discuss the appearance of the factor $\mathcal{M}_{\rm ks}$ in (3.28) in more detail. Recall that we introduced in (3.13) a special set of coordinates $S, \mathcal{T}^i, \mathcal{T}^a$ on $\mathcal{M}_{\rm ks}$. Imposing the orientifold constraints that in the large volume coordinates we have $b^S = b^i = v^\alpha = 0$ one shows that the $S, \mathcal{T}$ coordinates truncate as

$$\begin{aligned} C\mathcal{T}^1 \;&\to\; ie^{-\phi}\Big(v^1v^2 + \frac{1}{2}C_{ab}b^ab^b\Big)\,, \qquad & C\mathcal{T}^2 \;&\to\; \tfrac{1}{2}ie^{-\phi}\,, \\ C\mathcal{T}^a \;&\to\; -ie^{-\phi}b^a\,, & CS \;&\to\; iv^2v^S\,. \end{aligned} \tag{3.29}$$

In this evaluation C was used in the gauge associated to the coordinates $\mathcal{T}^i, \mathcal{T}^a$. It differs by a factor $2v^2$ from its large volume value $e^{-\phi}$ as imposed by its transformation property (3.12). We can now compare the orientifold truncations (3.29) with the definitions (2.15) and (3.26) of the $\mathcal{N}=1$ coordinates. The orientifold limit of the $CS, C\mathcal{T}^i, \mathcal{T}^a$ are precisely the imaginary parts of τ, G^a, T_S, T_1. Viewing the $\mathcal{N}=1$ coordinates as analytic continuation we can make the following identifications

$$\mathcal{T}^1 \;\rightarrow\; T_S\,, \qquad \mathcal{T}^2 \;\rightarrow\; \frac{1}{2}\tau\,, \qquad \mathcal{T}^a \;\rightarrow\; G^a\,, \qquad S \;\rightarrow\; 2T_1\,. \tag{3.30}$$

Using this map it is easy to check that also the $\mathcal{N}=1$ Kähler potential (3.27) for the scalars τ, G^a, T_S, T_1 can be identified with the Kähler potential K_D on $\mathcal{M}_{\rm ks}$ given in (3.15). This clarifies the fact that the special Kähler manifold $\mathcal{M}_{\rm ks}$ arises as the second factor in the $\mathcal{N}=1$ moduli space (3.28). In the next section we will discuss the holomorphic superpotential and use the duality map (3.30) to propose explicit expression for W arising from D3 instantons.

3.3 The D-instanton superpotential

In this section we propose a specific D-instanton superpotential for the Enriques orientifold. Since our main focus is the dependence of $W_{\text{D-inst}}$ on the moduli τ, G^a we will concentrate on the contribution proportional to e^{inT_S}. As seen in (3.26) only the complex coordinate T_S depends on the fields G^a and hence shifts as discussed in section 2.4. The imaginary part of T_S contains the volume form of the Enriques fiber modded out by the orientifold involution σ. If the corresponding four-cycle Σ can be extended to the F-theory picture such that it contributes to the D-instanton superpotential we expect a correction of the form

$$W_{\text{D-inst}} = \sum_n \Theta_n(\tau, G^a) e^{inT_S}\ . \tag{3.31}$$

In this expression we have also included multi-coverings of Σ labeled by n. A priory it is not clear that these will contribute and higher Θ_n might be zero.

We will now use our intuition from topological string theory on the Enriques Calabi-Yau and conjecture a possible form of $W_{\text{D-inst}}$. Recall that in section 3.1 we introduced a specific function $\Phi_{\rm B}(\mathcal{T}^1, \mathcal{T}^2, \mathcal{T}^a)$ encoding the lowest order degeneracies of D3, D1, D(-1) bound states on the Enriques Calabi-Yau. In such states, the D3 instanton wraps the Enriques fiber and couples to the complex coordinate $\mathcal{T}^1$, while the D1 branes wrap cycles in the E_8 lattice of the second cohomology and couple to complex coordinate $\mathcal{T}^a$. The D(-1) couple to the complex field $\mathcal{T}^2$ and appear in generic D3, D1, D(-1) bound states. Note that these are also the states which can appear in the instanton superpotential (3.31). More precisely, using the map (3.30) we identify the coordinates $\mathcal{T}^2, \mathcal{T}^a$ with the orientifold coordinates τ, G^a. The fiber volume appears in $\mathcal{T}^1$ which is identified with T_S. We now expand the function $\Phi_{\rm B}$ given in (3.17) in powers of e^{inT_S} as

$$\Phi_{\rm B}\Big(T_S, \frac{1}{2}\tau, G^a\Big) \;=\; \sum_n \theta_n(\tau, G^a)\, e^{inT_S}\,, \tag{3.32}$$

which defines the coefficients $\theta_n(\tau, G^a)$. Our proposal is that the G^a dependence of the D-instanton superpotential (3.31) arises through these functions $\theta_n(\tau, G)$. In other words, the superpotential arising due to D3 instantons on the Enriques fiber should take the form

$$W_{\text{D-inst}} = A_0 \sum_n \frac{c_n\ \theta_n(\tau, G^a)}{\eta^{10}(\tau)}\ e^{inT_S}\ , \tag{3.33}$$

where $\eta(\tau)$ is the standard eta-function and c_n are appropriate numerical coefficients. Unfortunately, without the complete F-theory picture we will not be able to check (3.33) directly and details might change in an explicit analysis. However, making contact to the discussion in section 2.4 we will discuss in the remainder of this section that the θ_n have the correct properties to ensure that $W_{\text{D-inst}}$ is a modular form of weight -1 in τ. Moreover, also the shifts of T_S given in (2.22) and (2.35) are appropriately canceled by shifts of θ_n as needed for consistency.

Let us finish this section with some remarks on the properties of the functions θ_n in (3.33). These can be determined explicitly by expanding the expression for Φ_{B} in the product representation (3.17) or the corresponding sum representation [33, 34]. It was shown in ref. [34] that Φ_{B} is an automorphic form of weight 4. Following the arguments of [54, 55] one deduces that the coefficient functions θ_n are Jacobi forms of weight 4 and index n, i.e. transform as given in (2.36) and (2.37) under modular transformations and B-shifts. In fact, in ref. [55] automorphic forms similar to Φ_{B} were constructed by combining appropriate Jacobi forms with the exponential e^{inT_S}. The precise form of θ_n is then determined by a lift of the modular coefficient functions such as (3.18). Instead of giving the explicit expressions for $\theta_n(\tau, G)$ we indirectly check some of their properties through a differential equation which they obey. In order to do that, we note that $\Phi_{\text{B}}(\mathcal{T})$ satisfies a wave equation of the form [33, 34][16]

$$2\frac{\partial^2 \Phi_{\text{B}}}{\partial \mathcal{T}^1 \partial \mathcal{T}^2} - C_D^{ab} \frac{\partial^2 \Phi_{\text{B}}}{\partial \mathcal{T}^a \partial \mathcal{T}^b} = 0\ . \tag{3.34}$$

This equation is readily translated into a condition on the functions $\theta_n(\tau, G)$ in (3.33). One finds

$$\left(in\frac{\partial}{\partial \tau} - \frac{1}{2} C^{ab} \frac{\partial^2}{\partial G^a \partial G^b} \right) \theta_n(\tau, G) = 0\, , \tag{3.35}$$

which is the higher-dimensional analog of the heat equation for theta-functions on an appropriate lattice. It also indicates that $\theta_n(\tau, G)$ are Jacobi forms as expected from the general discussion above. Since Φ_{B} and hence $\theta_n(\tau, G)$ are of weight 4 we conclude that the inclusion of the $\eta^{10}(\tau)$ factor ensures that $W_{\text{D-inst}}$ is of weight -1 as needed for (2.34). To actually show that $\theta_n(\tau, G)$ and $\eta(\tau)$ appear in the correct way in the conjectured superpotential (3.33) one might calculate $W_{\text{D-inst}}$ in a specific limit. In particular, it would be interesting to derive $W_{\text{D-inst}}$ in the orbifold limit using its heterotic dual.

4. Conclusions

In this paper we discussed the symmetries and non-perturbative corrections of the four-dimensional effective theory arising in type IIB orientifolds with O3 and O7 planes. We

[16] This is far from obvious in the product representation of Φ_{B}, but can be easily checked when writing Φ_{B} as a sum [33, 34].

studied both the Kähler potential and superpotential in the orientifold large volume limit for general $\mathcal{N}=1$ compactifications and later concentrated on a specific orientifold of the Enriques Calabi-Yau.

In our general analysis we first discussed the $\mathcal{N}=1$ Kähler potential including perturbative and non-perturbative α' corrections inherited from the underlying $\mathcal{N}=2$ theory. A subset of the non-perturbative α' corrections were shown to survive the orientifold large volume limit, since they depend on the scalars G^a arising from the NS-NS and R-R two-forms. They contribute to the Kähler potential in an explicitly calculable way, but do not alter the $\mathcal{N}=1$ chiral coordinates. It was argued that in order to ensure duality invariance of the α' corrections to the Kähler potential also contribution due to D(-1) and D1 branes have to be taken into account. In general, it seems hard to determine these corrections directly. We thus restrained ourselves to a brief discussion of candidate modular completions proposed for the underlying $\mathcal{N}=2$ theory. It would be interesting to derive these corrections explicitly by using heterotic-F-theory duality or be analyzing specific orbifold examples. Already the inclusion of the α' corrections will lead to interesting new phenomenological properties of these compactifications and a study of explicit examples is desirable.

From a phenomenological point of view the two-form scalars G^a have to be rendered massive in a vacuum. We have shown that this can be achieved by a potential induced by D3 instantons. More precisely, we have used the symmetries of the orientifold theory to argue that the two-form scalars arise through Jacobi forms in front of the D3 instanton contribution e^{inT} in the superpotential. These are generalizations of the well known theta-functions and depend on the dilaton-axion τ as modular parameter. Due to holomorphicity and modular invariance one might hope that the set of candidate Jacobi forms can be restricted to a finite set for a given example. Candidate forms should appear in topological string theory on the underlying Calabi-Yau manifold counting degeneracies of D1, D(-1) states on cycles which become singular in the orientifold background. Additional boundary conditions obtained in computations performed in specific limits of the theory might then fix the precise form of the D-instanton superpotential.

In the finial part of the paper we studied a specific example. We considered an orientifold of the Enriques Calabi-Yau. The kinetic terms of the four-dimensional $\mathcal{N}=1$ effective theory are determined in terms of a simple Kähler potential. We showed that the corresponding moduli of bulk moduli fields is a product of cosets. Interestingly, the reduction of the underlying quaternionic $\mathcal{N}=2$ geometry led to a Kähler manifold which can be identified with the original deformation space of the complexified Kähler structure of the underlying Calabi-Yau manifold times an $Sl(2,\mathbb{R})/\,\mathrm{U}(1)$ factor. This duality can be used in the study of the D-instanton superpotential on the Enriques Calabi-Yau. We mapped Jacobi forms known from topological string theory on the Enriques Calabi-Yau to the corresponding $\mathcal{N}=1$ orientifold. This lead to a conjecture of a specific D3-instanton superpotential. Unfortunately, explicit tests of this proposal are still missing and would involve a careful construction of an F-theory realization of the Enriques scenario. It would be also interesting to investigate other examples. Particularly, other K3 fibrations might allow to investigate similar questions, which can then be tested using string-string dualities.

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A. On the geometry of the Enriques Calabi-Yau

In this appendix we review some facts about the geometry of the Enriques Calabi-Yau and its cohomology lattice. Recall the cohomology lattice of the K3 surface is an even self-dual lattice with Lorentzian signature. Explicitly, it takes the form [62]

$$\begin{aligned} H^2(\mathbb{Z}) &\cong [\Gamma^{1,1} \oplus \Gamma_{E_8}(-1)]_1 \oplus [\Gamma^{1,1} \oplus \Gamma_{E_8}(-1)]_2 \oplus \Gamma^{1,1}_g \,, \\ H^0(\mathbb{Z}) \oplus H^4(\mathbb{Z}) &\cong \Gamma^{1,1}_s \,, \end{aligned} \tag{A.1}$$

where the inner products on the sublattices $\Gamma_{E_8}(-1)$ and $\Gamma^{1,1}$ are given by

$$-(C^{ab}) = -C_{E_8} \,, \qquad (C^{ij}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \,. \tag{A.2}$$

with $a, b = 1, \ldots, 8$ and $i, j = 1, 2$. Here C_{E_8} is the Cartan matrix of the exceptional group E_8. In other words, choosing a basis $\tilde{\omega}_K \in H^2(K3, \mathbb{Z})$ with $K = 1, \ldots, 22$ one has

$$\int \tilde{\omega}_K \wedge \tilde{\omega}_L = d_{KL} \,, \tag{A.3}$$

where d_{KL} equals to C_{ab} on elements of $\Gamma_{E_8}(-1)$ and C_{ij} on elements of $\Gamma^{1,1}$ and vanishes for all off-diagonal combinations in the lattice (A.1). Clearly, for the torus $\mathbb{T}^2$ we simply have the additional two-dimensional lattices $H^1(\mathbb{T}^2, \mathbb{Z})$ and $H^0(\mathbb{T}^2, \mathbb{Z}) \oplus H^2(\mathbb{T}^2, \mathbb{Z})$. In order to mod out the Enriques involution it is convenient to us an explicit algebraic realization of the $K3$ surface. For example, a $K3$ surface admitting such an involution can be obtained as a double covering of $\mathbb{P}^1 \times \mathbb{P}^1$ branched at the vanishing locus of a bidegree $(4, 4)$ hypersurface [30]. The Picard lattice of the resulting $K3$ has rank 18. Using this algebraic realization the action of the Enriques involution can be evaluated explicitly. Let us denote $(p_1, p_2, p_3) \in H^2(K3, \mathbb{Z})$ corresponding to the three terms in (A.1) and abbreviate $p_4 \in H^0(K3, \mathbb{Z}) \oplus H^4(K3, \mathbb{Z})$ as well as $p_5 \in H^1(\mathbb{T}^2, \mathbb{Z})$. The $\mathbb{Z}_2$ involution on the Enriques Calabi-Yau acts on the elements p_i as [29][17]

$$|p_1, p_2, p_3, p_4, p_5\rangle \to e^{\pi i \delta \cdot p_4} |p_2, p_1, -p_3, p_4, -p_5\rangle \,, \tag{A.4}$$

[17] The effect of the phase factor on the type II side was interpreted as turning on a Wilson line [29].

where we denoted $\delta = (1, -1) \in \Gamma_s^{1,1}$. It it now straight forward to deduce the cohomology of the Enriques Calabi-Yau

$$H^2(Y_E, \mathbb{Z}) \cong \mathbb{Z} \oplus \Gamma^{1,1} \oplus \Gamma_{E_8}(-1)\,, \tag{A.5}$$

$$H^3(Y_E, \mathbb{Z}) \cong \left(\Gamma^{1,1} \oplus \Gamma_{E_8}(-1) \oplus \Gamma_g^{1,1}\right) \oplus \left(\Gamma^{1,1} \oplus \Gamma_{E_8}(-1) \oplus \Gamma_g^{1,1}\right), \tag{A.6}$$

where elements of $H^2(Y_E, \mathbb{Z})$ are of the form $p_1 + p_2$ while elements of $H^3(Y_E, \mathbb{Z})$ are of the form $p_5 \wedge (p_1 - p_2)$. One thus shows that the dimensions $h^{(p,q)}$ of the cohomologies $H^{(p,q)}(Y_E)$ are $h^{(1,1)}(Y_E) = h^{(2,1)}(Y_E) = 11$. The Enriques Calabi-Yau is shown to be self mirror [29]. The two eleven-dimensional moduli spaces of complex and Kähler structure deformations are identified with the coset (3.5) mod the symmetry group $Sl(2,\mathbb{Z}) \times O(10,2,\mathbb{Z})$ as discussed.

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